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A Comparison of Classical Analytic Theories for the Motion of Artificial Satellites

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Motivated by the heavy reliance upon the analytic orbit theory in orbit determination operations, a comparison study is performed for classical analytical theories of artificial satellite motion about an oblate Earth. Comparison results for each theory are produced for a number of representative satellites of current or past interest which proved amenable to analytic theory application. The uniformity of these results has significant implications for current and future mission operations and planning activities.

I. Introduction

A MAJOR task associated with spacecraft project support is the regular process of orbit estimation and prediction. Explicit to this process is the refinement of orbital parameters from the use of observational data judiciously combined with a mathematical theory of artificial satellite motion. (The term theory is used in a broad sense as a set of mathematical expressions which can be evaluated or invoked to produce an orbital position estimate. Therefore, it embraces many different methods.) Owing to the capricious nature of the total estimation algorithm, the performance of different orbital representations, each with its own fitted initial conditions, is apt to vary. However, it is not unreasonable to assume that despite these estimation vagaries, the more accurate theory will generally yield the most reliable results. Therefore, a topic with immense practical implications is the performance of selected, competitive theory (each with approximately the same computational speed) following a data reduction cycle. Although practical considerations make such direct in-the-field accuracy comparisons beyond the scope of this study, the results presented in this report will indirectly address this area in a mathematically precise and dynamically meaningful manner for orbit theories of a particular generic type, the analytic theory.

An analytic theory or method of analytic perturbations represents a set of analytical expressions which, when properly initialized and evaluated, yields an estimate of the satellite's disturbed orbit varying under the influence of the perturbative forces. Herein lies the computational advantage of the analytic theory. Namely, if the spacecraft position and velocity $\bar{R}_0$, $\dot{\bar{R}}_0$ at time t_0 are known, then $\bar{R}_t$, $\dot{\bar{R}}_t$ at arbitrary time t are obtainable with a single evaluation of the analytical expressions. Thus, it is no more computationally burdensome to obtain the orbital elements at $t \gg t_0$ than $t \approx t_0$.

The desirable computational advantage of the analytic theories is not possessed by an alternative class of orbit techniques known as special perturbation methods, which are oriented toward computational algorithms rather than time-dependent analytical expressions for perturbed motion. Most illustrative of the techniques of this class are the numerical integration algorithms applied to the differential equations of motion. Although more extensive perturbations can be treated with these methods (explaining their widespread use for special applications), the step-by-step approach can be time-consuming if $t \gg t_0$ relative to the step size, while the solution itself is degraded by an error that accumulates with the number of steps taken. In contrast, the analytic theories are not burdened with this cumulative error growth endemic to the numerical techniques of the special perturbation methods. Rather, the analytic theories to be discussed have distinctive errors resulting from either the inadequacy of the analytical solution itself or truncation of the analytical expressions. This fact causes distinct variations in the accuracy of the solutions which are directly observable in the proper experimental context.

The analytic theories examined in this paper are the classical theories[1] of D. Brouwer, a modified Brouwer, i.e., Brouwer-Lyddane and Cohen, and J. Vinti. Each theory describes satellite motion about a planet, whose deviations from sphericity are represented with the potential of the respective theories expanded in spherical harmonics with zonal harmonic perturbations through J_5.

Notwithstanding exclusions to analytic theory application, a large class of satellites is admissible to the domain of analytic theory application. The widespread usage of the analytic theories when applicable has proven to be indispensable and imperative to the practical implementation of a workable large-scale orbit determination support facility. Special perturbation methods prove to be a luxury too expensive to sustain for all supported satellites. Therefore, it is not surprising that a significant portion of satellites have been and are currently being supported by an analytic theory. Some sample satellites are: NOAA-4 (1974-89A), NOAA-3 (1973-86A), SAS-3 (1975-37A), OAO-3 (1972-65A), UK-5 (1974-77A), NIMBUS-5 (1972-97A), NIMBUS-6 (1975-52A), OSO-8 (1975-57A), ISIS-1 (1969-09A), and ISIS-2 (1971-24A).

Among previous studies which have addressed the question of direct analytic theory comparison for a limited set of orbits are Bonavito[2] and Lubowe.[3] Analogous to these studies, this report will produce, in part, comparative data on the accuracy of each candidate method to the *reference solution*§ of the

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§The numerical integration of the differential equations of motion with perturbing accelerations caused by the disturbing influence of the zonal harmonics J_2 through J_5 only; with a 12th-order Cowell-Störmer predictor-corrector with a fixed step size of 1/200th of an orbital period.[1]

appropriate orbit dynamical problem. However, this study expands the scope of comparisons to include numerous cases of current analytic theory support. In addition, the cases presented offer a span of results where the physical significance of the conceptual differences can be assessed. These differences will be addressed in this study. It would be beneficial for operational purposes and theoretical concerns if a uniformity of comparison results could be realized. For though not conclusive proof of better in-the-field performance of one theory over another, the implications of such a finding, based upon pure orbit comparison results, would be inescapable, and further examination and direct experimentation certainly merited.

II. Selected Analytical Orbit Theories

This paper concerns itself with comparisons of the relative approximation accuracies of two classical first-order[4] artificial satellite theories—Brouwer, a modified Brouwer, i.e., Brouwer-Lyddane, and Vinti. Both theories represent general perturbation methods which address themselves to the problem of artificial satellites, which adhere to closed body-centric orbits and are disturbed by perturbations arising from the nonsphericity of the central body.

These two classical analytical orbit theories have substantially different conceptional approaches to providing an approximation to the solution of the problem they mutually address. Indeed, no small confusion exists in the community which employs these theories as to how these conceptional differences reflect upon the relative approximation merits of these respective theories. This consideration provides the motivation for the comparison study.

Brouwer[5] derived a first-order perturbation solution expressing the secular, short- and long-periodic variations in the motion of an aritificial satellite about an oblate planet, with the Earth's potential approximated by an expansion in spherical harmonics of the form

$$V=\frac{\mu}{r}\left[1-\sum_{n=2}^{5}\left(\frac{R_E}{r}\right)J_nP_n(\sin\delta)\right]$$

J_2 through J_5 are zonal harmonic coefficients, μ the gravitational constant, r the radial magnitude, δ the satellite declination angle, P_n the Legendre polynomial of order n, and R_E the mean equatorial Earth radius. The short-period terms contain linear multiple combinations of the mean argument of perigee and the mean true anomaly in their arguments, with the arguments of the long-periodic terms appearing as integer multiples of the mean argument of perigee.

Brouwer obained separation of all the periodic terms by adapting Von Zeipel's[6] technique to modify Delaunay's method for calculating the coefficients of the periodic terms through a succession of canonical transformations. Delaunay variables were introduced in order to simplify the canonical expressions for the equations of motion. Brouwer developed the periodic terms to $0(J_2)$ and obtained the secular variations to $0(J_2^2)$. The resultant formulae are piecewise continuous with singularities existing for certain values of the eccentricity and inclination which occur as poles in the algebraic expressions. Thus, the equations are valid, except in the regions for which $e''=0$, $i''=0$ deg, and $1\text{-}5\cos^2 i''=0$, i.e., $i''=63.43$ deg, the critical inclination.¶

Lyddane[7] introduced Poincaré variables and reformulated Brouwer's expressions so as to remove the poles, and thus the singularities arising from small eccentricities or inclinations in the Brouwer theory. The expressions employed[8] are those derived by Siry[9] from Lyddane's paper.

Vinti[10] obtains an accurate first-order[4] approximation of the effect of oblateness on the orbit of an artificial satellite by solving a dynamical problem in oblate spheroidal coordinates.

¶The double-primed elements are the mean elements of the theory.

He determines a potential function satisfying Laplace's equation that leads to separability of the Hamilton-Jocobi equation, and is adjusted so as to adhere to the geopotential exactly through the third zonal harmonic. Vinti presented the relationships existing between the zonal coefficients which reduce the solution of the motion of a particle in the field of the Vinti potential to a separable problem as

$$J_4=-J_2^2+J_3^2/J_2 \tag{1a}$$

$$J_5=-2J_3(J_2-\tfrac{1}{2}J_3^2/J_2) \tag{1b}$$

However, the best-known values of the geodetic coefficients do not exactly satisfy these relations. It is interesting to note that the potential implies a geodetic inference of the density distribution in the geoid, and thus suggests a first-order hypothesis for the mass distribution within the Earth. Vinti develops his solution to $0(J_2^3)$ in the secular terms and to $0(J_2^2)$ in the periodic terms. The Vinti solution contains no singularities for small eccentricities or inclinations, or the critical inclination.

There exist few dynamical systems in theoretical mechanics which provide for separability of the Hamilton-Jacobi equation. Thus, approximate solutions are often developed which employ first-order perturbation methods. Such techniques are applicable when the disturbing forces are small and a solution can be obtained for the unperturbed dynamical system.

Consider the total Hamiltonian H of a dynamical system, which yields a nonseparable Hamilton-Jacobi equation, to represent perturbed motion for an undisturbed Hamiltonian H_0, which leads to a separable solution. Then, H may be expressed by

$$H=H_0+\mathcal{H}$$

with

$$\mathcal{H}=H-H_0+0(k)$$

where k is a small parameter. The Hamilton-Jacobi equation for this dynamical system may be amenable to first-order perturbation theories in order to provide an approximate solution of the true Hamiltonian H.

Brouwer makes the natural choice of the potential $V_0=\mu/r$, which leads to a solution of the undisturbed Hamiltonian H_0, which describes the Keplerian elliptic orbit of an artificial satellite. Brouwer develops a first-order perturbation theory for the solution of the true Hamiltonian with the perturbed Hamiltonian, functionally expressed by

$$\mathcal{H}_B=H-H_0+0(J_2,J_3,J_4,J_5)$$

where J_2, J_3, J_4, and J_5 are the zonal harmonics of the geopotential.

Vinti determines a potential such that the undisturbed Hamiltonian H_0 in oblate spheroidal coordinates has the functional relation

$$H\equiv H_0(\mu,J_2,J_3)$$

for which he obtains separability of the Hamilton-Jacobi equation. To provide for such separability, Vinti imposes relationships, Eq. (1), for the zonal harmonic coefficients, for which the calculated values for his J_4 approximates two-thirds of the empirical values determined for J_4, and approximately 1/100th of the accepted value for J_5.

The unperturbed Hamiltonian H_0 of the Vinti dynamical problem includes the oblateness effect of J_2 and J_3, whereas the unperturbed Hamiltonian H_0 of the Brouwer solution does not. Since Vinti demonstrated that his potential could be adjusted so as to exactly match the geopotential for zonal harmonic coefficients up to and including J_3, the higher order

Table 1 Test case orbital elements

	a, km	e	i, deg	H_p^a, km	H_a^a, km	P, min
ISIS-1	8422.304	0.174536	88.427	568.35	3522.81	128.238
NIMBUS-6	7477.181	0.000823	99.958	1092.86	1105.17	107.242
NOAA-3	7882.807	0.000582	101.980	1500.05	1509.23	116.086
OAO-3	7122.446	0.000670	35.008	739.51	749.05	99.701
OSO-8	6914.110	0.001640	32.977	524.61	547.27	95.359
SAS-3	6890.419	0.000581	2.991	508.25	516.25	94.869
UK-5	6909.801	0.003160	2.874	509.79	553.48	95.270

[a] H_p = perigee height, H_a = apogee height.

to which he develops his solution is indicative of a more accurate approximation to the solution of the oblateness effects upon the motion of an artificial satellite. However, the relationships required for the computation of the J_4 and J_5 zonal effects in the Vinti dynamical problem imply variations in the relative accuracy of the two orbital theories as a function of variations in the orbital geometries. Cognizance is taken of the J_4 zonal contribution to the secular variations, and that both J_4 and J_5 contribute to the long-periodic effects. Thus, there exists motivation for a comparison study in providing information for determining the degree that these constraints could have in possibly degrading the Vinti solution relative to the Brouwer solution.

III. Results

The orbital shape and orientation parameters for the test cases are presented in Table 1. Figures 1-7 represent the position deviations between the analytic theories and the reference solution, decomposed into the radial, cross-track, and along-track components for each of the seven test cases. Note that the radial and cross-track scales are in meters, whereas the dominant along-track departures are given in kilometers. The comparisons span over two weeks. Table 2 presents the maximum absolute deviations from the reference solution for each theory and test case. All tabular entries are in kilometers.

It is noted that the Brouwer theories were initialized with mean elements obtained from an extension of the algorithm recommended by Walter[11] for osculating element reduction. This extension, necessitated by algorithmic divergence for elements in a neighborhood of the Brouwer singularities, entailed the rigorous employment of the osculating-to-mean element Jacobian. When extended, Walter's algorithm provides the Brouwer theories with practically errorless initialization in all cases.

Another method which removes the apparent instability of the iterative osculating-to-mean element conversion for the Brouwer/Brouwer-Lyddane theories is to translate the iteration from mean Keplerian space to mean Cartesian space. The six Cartesian elements are denoted by the symbols $X_1, \ldots, X_6$, and the corresponding osculating Cartesian elements by $Y_1, \ldots, Y_6$. Employing the iterative algorithm

$$X_i^{(j+1)} = X_i^{(j)} + (Y_i - Y_i^{(j)})$$

now involves the additional step of a two-body conversion of the updated mean Cartesian elements to Keplerian mean elements to reinitialize the Brouwer/Brouwer-Lyddane theories at each iterative step.

The two-week results reflect accuracies of each analytic theory's approximation of the secular oblateness variations.

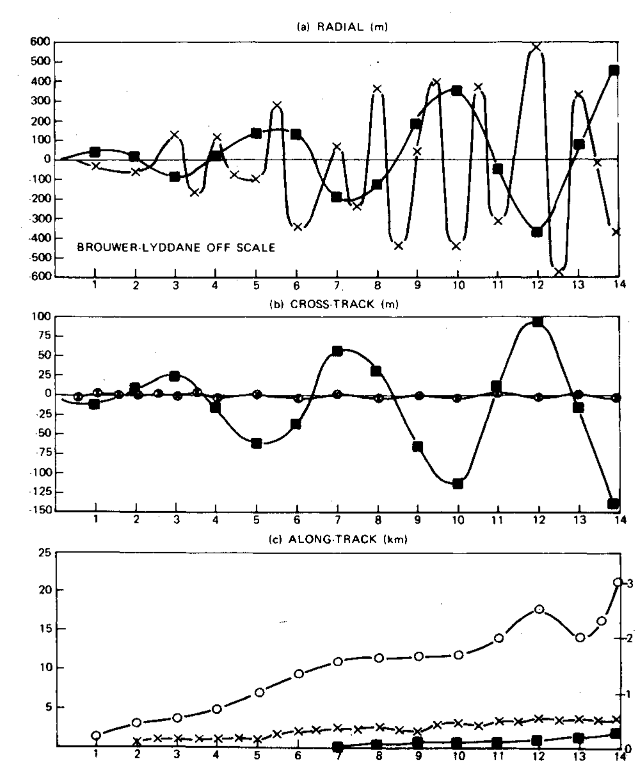

Fig. 1 ISIS-1 analytic theory orbit differences (× —Brouwer, ○ —Brouwer-Lyddane, ■ —Vinti).

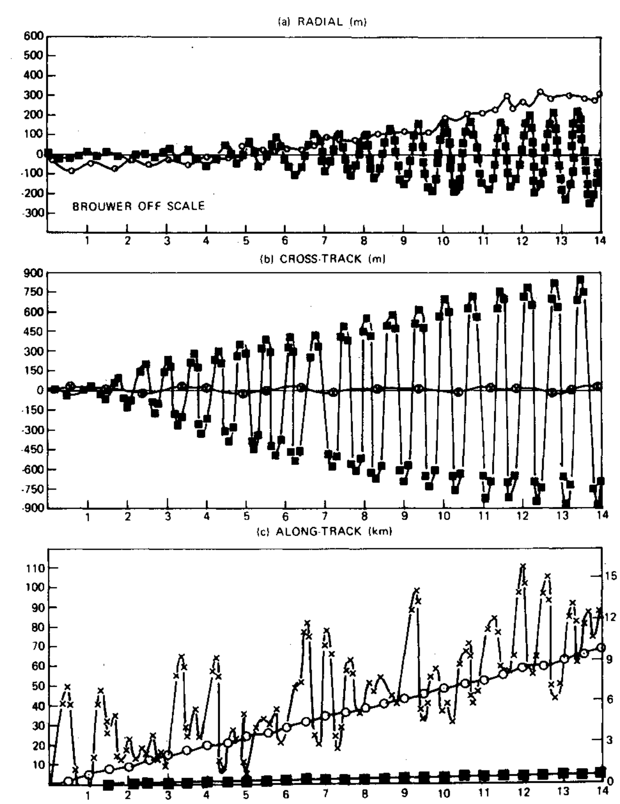

Fig. 2 NIMBUS-6 analytic theory orbit differences (× – Brouwer, ○ —Brouwer-Lyddane, ■ —Vinti).

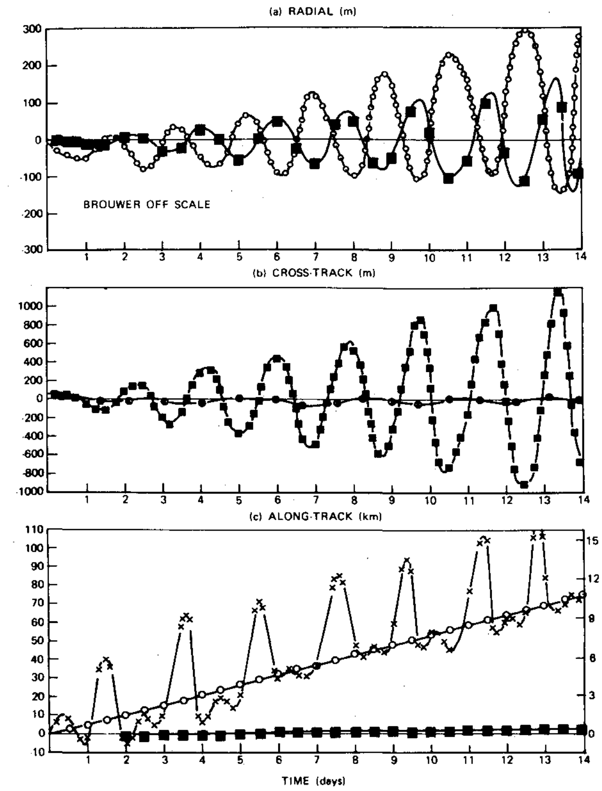

Fig. 3 NOAA-3 analytic theory orbit differences (×—Brouwer, ○—Brouwer-Lyddane, ■—Vinti).

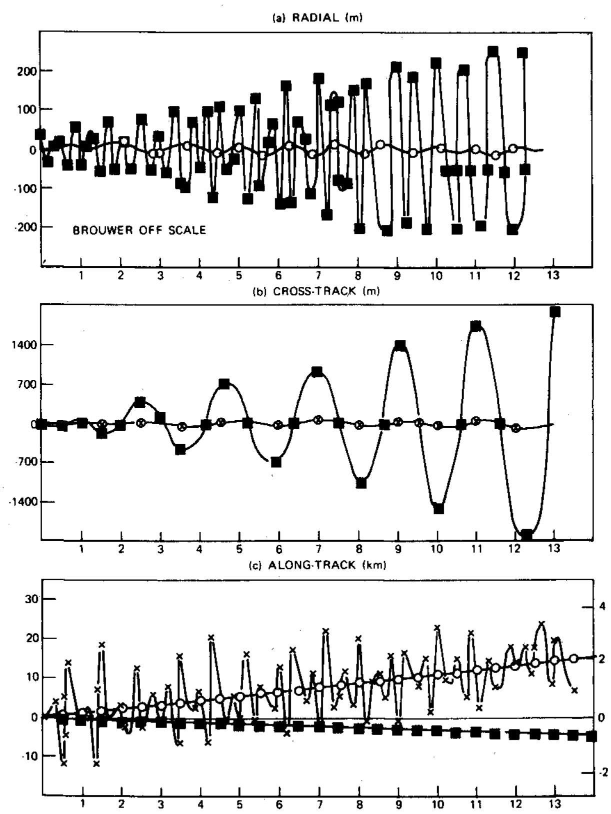

Fig. 4 OAO-3 analytic theory orbit differences (×—Brouwer, ○—Brouwer-Lyddane, ■—Vinti).

IV. Implications of Results

All of the two-week comparison results manifestly demonstrate the superior approximation of the Vinti solution with respect to the Brouwer/Brouwer-Lyddane solutions for representing the oblateness effects of the Earth upon the orbit of an aritificial satellite. They reflect the improvement in the solution exhibited by Vinti's development of his orbit theory to $0(J_2^3)$ in the secular terms, as opposed to Brouwer's development of the secular terms of his theory to $0(J_2^2)$; this was further corroborated by the results obtained from mean element comparisons. These results confirm the dominance of the higher order J_2 effects relative to J_4 in contributing to along-track deviations. The results further indicate the variation of the relative improved approximation of the Vinti solution as interrelated to various orbital geometries; however, the improved approximation of the Vinti solution in the total position differences generally remains substantial.

The effect the approximate calculated value of J_4 has upon cross-track errors for the Vinti solution is also graphically illustrated in Table 2 and the figures presented. While the cross-track error normally represents only a small contribution to the total error, it does reflect the effect this approximation has on the Vinti solution when compared with the Brouwer-Lyddane solution which contains the full value of J_4. This is caused by the appearance and significance of J_4 in the calculation of the secular and long-period perturbations to the longitude of the ascending node. This provided the motivation for an enhancement of the Vinti solution to include the contribution to the Vinti reference orbit of the residual or neglected component of J_4, which is described in the Appendix.

While test cases over a span of permutations of orbital elements were made, the cases presented in this study provide all of the information content gained from this experience; in addition, the cases chosen for this study are at present operationally supported by analytic theories.

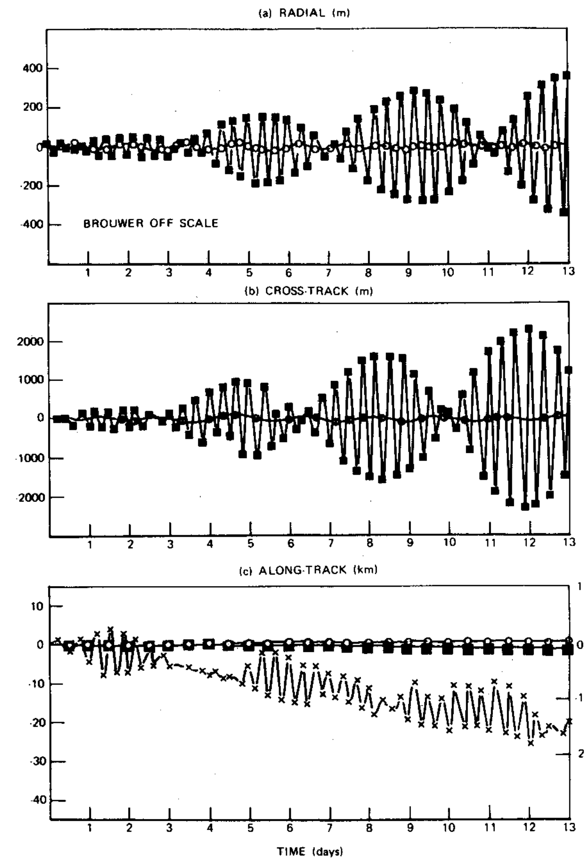

Fig. 5 OSO-8 analytic theory orbit differences (×—Brouwer, ○—Brouwer-Lyddane, ■—Vinti).

V. Conclusions

1) How any orbital theory representation will respond in a least-squares estimation sense to observational fitting cannot

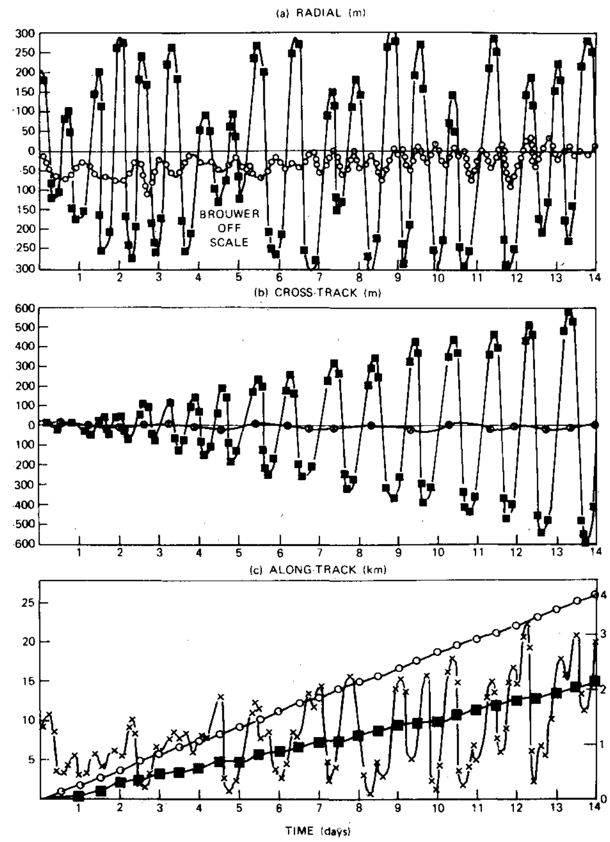

Fig. 6 SAS-3 analytic theory orbit differences (×—Brouwer, ○—Brouwer-Lyddane, ■—Vinti).

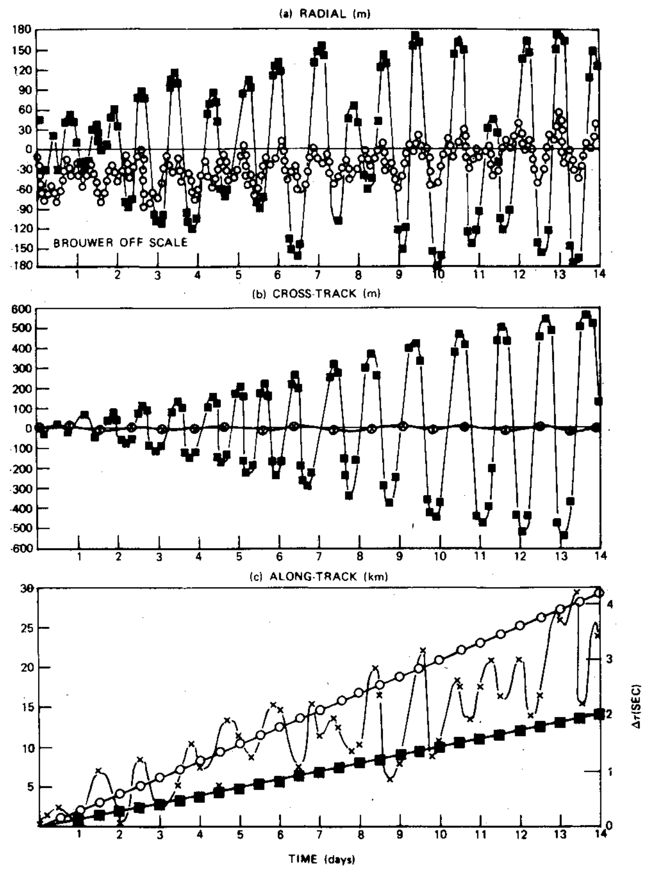

Fig. 7 UK-5 analytic theory orbit differences (×—Brouwer, ○—Brouwer-Lyddane, ■—Vinti).

be universally ascertained. It is not unreasonable to assume, however, that the more accurate representation of two analytical theories, which address a solution to the same artificial satellite motion problem, will for the most part yield the most reliable results. As a consequence of the rather substantially improved orbital representation that can be realized by the Vinti orbit theory (as demonstrated in this paper), and in view of the fact that most of the satellites supported by analytical theories are nearly circular and Vinti requires no special treatment for such orbital geometries, its potential for improvement in the operational support environment may be realized.

2) Predicated upon the results obtained, there exists no ambiguity as to which of these theories should prove to be more applicable in a mission analysis support environment.

Table 2 Maximum absolute orbital differences

Satellite		Total position, km	Radial, km	Cross-track, km	Along-track, km
	B	4.978	0.665	0.002	4.978
ISIS-1	B-L	21.072	2.892	0.002	21.072
	V	2.092	0.445	0.145	2.090
	B	111.572	19.111	0.017	110.105
NIMBUS-6	B-L	67.476	0.329	0.017	67.475
	V	4.903	0.205	0.981	4.872
	B	117.637	2.769	0.013	116.605
NOAA-3	B-L	76.014	0.293	0.022	76.010
	V	4.130	0.123	1.282	4.022
	B	23.405	4.844	0.061	23.335
OAO-3	B-L	14.370	0.021	0.354	14.370
	V	4.809	0.294	2.182	4.629
	B	26.596	4.824	0.048	26.592
OSO-8	B-L	0.902	0.014	0.058	0.902
	V	2.697	0.368	2.347	1.379
	B	21.758	12.971	0.048	20.071
SAS-3	B-L	24.037	0.078	0.056	24.037
	V	13.625	0.359	0.646	13.607
	B	25.987	3.268	0.056	25.864
UK-5	B-L	26.951	0.071	0.059	26.951
	V	12.517	0.182	0.559	12.507

Table A1 Maximum absolute orbital differences

	Radial, m		Along-track, km		Cross-track, m	
Satellite	Without Vinti residual 4th	With Vinti residual 4th	Without Vinti residual 4th	With Vinti residual 4th	Without Vinti residual 4th	With Vinti residual 4th
SAS-3	359.6	361.7	15.1	1.6	646.2	6.7
NOAA-3	130.3	130.7	4.6	2.8	1025.5	31.6
OSO-8	368.2	363.7	1.4	0.9	349.3	19.4

3) The algorithm presented in the Appendix is shown to provide an enhancement to Vinti for an improved accounting of the J_4 zonal harmonic effect.

4) The evidence is incontrovertible that the Vinti theory proves to be more accurate, and applicable to a variety of artificial satellites. These considerations should initiate and facilitate the direct assessment of the full potential of the Vinti theory.

Appendix

By observing the results, it can be seen that while the Vinti theory has a much lower along-track position error than Brouwer or Brouwer-Lyddane, Vinti has a modestly large cross-track error relative to the other two theories. Upon analysis of the osculating Keplerian element comparisons, it was hypothesized that the part of the fourth harmonic not included in Vinti's theory could be the cause of the large cross-track error. In order to test this hypothesis, comparisons were made for several cases matching the reference solution to a Cowell numerical integration identical to the reference solution, except with the fourth harmonic defined by the Vinti potential as

$$J_4 = -J_2^2 + J_3^2/J_2 \qquad \text{(A1)}$$

which is approximately 70% of the accepted value. These comparisons nearly reproduce the Vinti vs reference solution results, thus lending credence to the hypothesis that the residual fourth harmonic was the source of the Vinti cross-track errors.

Having established the probable impact of a partial J_4, a full accounting of J_4 can be included by employing a first-order perturbation approximation[12] for the Vinti dynamical system, which now experiences a perturbed Hamiltonian, functionally represented by

$$\mathcal{H}_v = H - H_0(\mu, J_2, J_3) + 0(\sigma_4)$$

The solution of this Hamiltonian is accomplished with virtually indiscernible computational cost.[13]

The residual fourth harmonic is defined as the difference, σ_4, of the accepted J_4 and Vinti's value, Eq. (1); i.e.,

$$\sigma_4 = J_4 + J_2^2 - J_3^2/J_2$$

A few representative examples follow showing the reduction of the cross-track (the same order as Brouwer and Brouwer-Lyddane) and along-track position errors when applied to the SAS-3 (low e, low I), NOAA-3 (low e, high I), and OSO-8 (low e, medium I) satellites.**

Table A1 gives the maximum absolute orbital differences for the three satellites with and without the residual fourth harmonic. It can be seen that the radial error for all three satellites remained virtually unchanged.

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**See Table 1 for precise elements.